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Classical limit of the harmonic oscillator Wigner functions in the Bargmann representation

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Abstract. Writing the Wigner functions $W(\psi_n, \psi_m)(p, q)$ of any pair of harmonic oscillator eigenstates ψ_n, ψ_m in the Bargmann representation, a direct and detailed proof is given of their convergence (in the sense of distributions) to $\delta(p^2 + q^2 - A)\,e^{i(m-n)\phi}$ at the classical limit $n \to \infty$, $\hbar \to 0$, $n\hbar \to A$, $m - n$ fixed, $\phi = \arctan(p/q)$.

1. Introduction

The problem of the classical limit of the matrix elements of the quantum observables has recently drawn much attention, both in the chaotic (see [CdV], [Com–Rob], [DEGI], [HMR], [Zel1]) as well as in the integrable and quasi-integrable case (see [Bel–Vit], [Cha], [DEGH], [Gra–Pau], [Zel2]). A well known convenient way to study this problem is to represent the above matrix elements through the Wigner function formalism (see e.g. [Bal–Jen], [Bar], [Ber-Bal], [Ber], [Gro], [HOSW], [Raj], [Tak], [Vor]): given two quantum states represented by the vectors $(\psi, \varphi) \in L^2(\mathbb{R}^l)$, the corresponding Wigner function $W(\psi, \varphi)(q, p)$ is defined as follows (see e.g. [Wig])

$$W(\psi, \varphi)(q, p) = \int_{\mathbb{R}^l} e^{i\langle p,x\rangle} \psi\left(x - \tfrac{1}{2}q\right) \overline{\varphi\left(x + \tfrac{1}{2}q\right)}\, dx \tag{1.1}$$

(here $\langle p, x\rangle = p_1x_1 + \cdots + p_lx_l$, $dx = dx_1 \cdots dx_l$). The Wigner function is manifestly defined on the phase space $\mathbb{R}^{2l}$; given any classical observable $f(p, q)$ and the corresponding quantum observable F obtained through canonical quantization, it relates the matrix elements of F to f in the following way:

$$\langle\psi, F\varphi\rangle = \int_{\mathbb{R}^{2l}} W(\psi, \varphi)(p, q) f(p, q)\, dp\, dq. \tag{1.2}$$

If $\psi = \{\psi_n(\hbar)\} : n = (n_1, \ldots, n_l), n_k \in \mathbb{N}$ is a sequence of eigenstates of a Schrödinger operator S generated by canonical quantization of an integrable classical Hamiltonian, it is expected (for the case $m = n$ the assertion is already strongly supported by known stationary phase arguments; see [Ber], [Ber–Bal]) that the corresponding sequence of matrix elements $\langle\psi_n, F\psi_m\rangle$ converge (in the sense of distributions; see below) to $\delta(H(p_1, \ldots, p_l; q_1, \ldots, q_l) - E(A_1, \ldots, A_l))\, e^{i\langle(m-n),\phi\rangle}$ at the classical limit $n_k \to \infty$, $\hbar \to 0$, $n_k\hbar \to A_k$, $k = 1, \ldots, l$. Here $(A_1, \ldots, A_l; \phi_1, \ldots, \phi_l)$ are the action-angle variables of the integrable Hamiltonian $H(p_1, \ldots, p_l; q_1, \ldots, q_l)$, related to the $(p, q) = (p_1, \ldots, p_l; q_1, \ldots, q_l)$ coordinates by the canonical transformation $(A, \phi) = \mathcal{C}(p, q)$, so that $E(A_1, \ldots, A_l) = H(\mathcal{C}^{-1}(A, \phi))$, and $\langle(m - n), \phi\rangle = (m - n)_1\phi_1 + \cdots + (m - n)_l\phi_l$.

An important particular case is represented by the harmonic oscillators, namely

$$H(p,q) = \tfrac{1}{2}\sum_{k=1}^{l}(p_k^2 + \omega_k^2 q_k^2). \tag{1.3}$$

In this case $E(A_1, \cdots, A_l) = \omega_1 A_1 + \cdots + \omega_l A_l$ and, if the frequencies $\omega_k : k = 1, \ldots, l$ are rationally independent, the above limit is simply (always in a distributional sense)

$$\lim_{\substack{n,m\to\infty,k=m-n\\ n\hbar\to A,\hbar\to 0}} W(\psi_n, \psi_m)(p,q)$$
$$= \delta(p_1^2 + \omega_1^2 q_1^2 - A_1)\cdots\delta(p_l^2 + \omega_l^2 q_l^2 - A_l)\cdot \mathrm{e}^{\mathrm{i}\langle(m-n),\phi\rangle}. \tag{1.4}$$

Inserting in (1.2) and integrating in polar coordinates (equivalently action-angle variable)

$$q = \sqrt{A}\cos\phi \qquad p = \sqrt{A}\sin\phi$$

this yields

$$\langle\psi_n, F\psi_m\rangle \to \int_0^{2\pi}\cdots\int_0^{2\pi} f(\sqrt{A_1}\cos\phi_1, \sqrt{A_1}\sin\phi_1, \ldots, \sqrt{A_l}\cos\phi_l, \sqrt{A_l}\sin\phi_l)$$
$$\times \mathrm{e}^{\mathrm{i}\langle(m-n),\phi\rangle}\,\mathrm{d}\phi_1\cdots\mathrm{d}\phi_l. \tag{1.5}$$

This last integral is nothing more than the k-Fourier coefficient $f_k(A)$, $k = (m-n)$, of the observable f over the l-torus labelled by A so that we have the identification

$$\lim_{\substack{n,m\to\infty,k=m-n\\ n\hbar\to A,\hbar\to 0}} \langle\psi_n, F\psi_m\rangle = f_k(A). \tag{1.6}$$

This formula, already implicitly contained in the treatise of Landau and Lifshitz ([Lan–Lif], section 48), is furthermore the cornerstone of recent convergence proofs ([Bel–Vit, Gra–Pa, DEGH]) of the quantum Rayleigh–Schrödinger perturbation theory around a system of non-resonant harmonic oscillators to the corresponding canonical perturbation theory at the above classical limit. It has, however, been proved only for polynomial perturbations $f(p,q)$. Therefore the detailed proof of (1.4), besides its intrinsic interest (it can be, furthermore, remarked that the identification of the limit of the Wigner function between different eigenstates has never been made explicit so far, at least to my knowledge), has the immediate application of extending (1.6), and hence the statement on the classical limit of Rayleigh-Schrödinger perturbation theory, to any smooth observable $f(p,q)$.

The convergence result will be proved by writing the Wigner function in the Bargmann representation [Bar]. An alternative approach could be tried using the coherent states (see [DB], [DHI]).

The presentation will be as follows. In the forthcoming section 2. I begin by recalling the construction of the Wigner function in the Bargmann representation, following essentially [Fol], and in section 3 I describe the convergence proof.

2. The Wigner function in the Bargmann representation

Let us begin by recalling some well known results on the Fock–Bargmann representation that we will use (see [Bar]); to simplify the exposition we consider a system of oscillators with unit masses and frequencies.

Given the standard canonical coordinates $(q,p) \in \mathbb{R}^{2n}$ we introduce the complex canonical coordinates $(z_k, \bar{z}_k)$ (Bargmann variables)

$$z_k = \frac{q_k + \mathrm{i}p_k}{\sqrt{2}} \qquad \bar{z}_k = \frac{q_k - \mathrm{i}p_k}{\sqrt{2}} \qquad k = 1, \ldots, n.$$

Remark that $\{z_j, \bar{z}_k\} = \mathrm{i}\delta_{jk}$, and that the canonical quantization of these variables yields the creation and annihilation operators of the harmonic oscillator

$$a_k^+ = \frac{Q_k + \mathrm{i}P_k}{\sqrt{2}} \qquad a_k = \frac{Q_k - \mathrm{i}P_k}{\sqrt{2}}$$

where Q_k and P_k are the standard position and momentum operators in $L^2(\mathbb{R}^n)$. The basic results concerning the Bargmann representation of quantum mechanics can be summarized as follows (see e.g. [Bar]).

Proposition 2.1. Let $\mathcal{F}$ be the space of all entire holomorphic functions on $\mathbb{C}^n$ defined as

$$\mathcal{F} = \Bigg\{ f : \mathbb{C}^n \to \mathbb{C} f \text{ entire holomorphic function such that } \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^{2n}} |f(z)|^2 \,\mathrm{e}^{-|z|^2/\hbar} \prod_{k=1}^{n} \mathrm{d}z_k \,\mathrm{d}\bar{z}_k < +\infty \Bigg\}.$$

The scalar product is

$$\langle f, g\rangle_{\mathcal{F}} = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^{2n}} f(z)\overline{g(z)}\,\mathrm{e}^{-|z|^2/\hbar}\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

so that $|z|^2 = \langle z, \bar{z}\rangle$. Here $z_k = x_k + \mathrm{i}y_k$ and hence $\mathrm{d}z_k\,\mathrm{d}\bar{z}_k = \mathrm{d}x_k\,\mathrm{d}y_k$.
For any $\varphi \in L^2(\mathbb{R}^n)$, the Bargmann transform

$$(B\varphi)(z) \equiv f(z) = (\pi\hbar)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-(1/2\hbar)(z^2 + x^2 + 2\sqrt{2}zx)} \varphi(x)\,\mathrm{d}x \tag{2.1}$$

is a unitary map between $L^2(\mathbb{R}^n)$ and the Fock–Bargmann space $\mathcal{F}$ and the following unitary equivalences hold,

$$\begin{cases} Ba_k^+ B^{-1} = Y_k \\ Ba_k B^{-1} = Z_k \end{cases} \qquad \begin{cases} BQ_k B^{-1} = Z_k - Y_k \\ BP_k B^{-1} = Z_k + Y_k \end{cases}$$

where Y_k and Z_k are the maximal multiplication operator by z_k and the maximal differentiation operator generated by $\hbar\partial_{z_k}$ in $\mathcal{F}$, respectively.

Let us recall the definition of the Wigner function in the Bargmann representation. Given any wavefunction $\psi(q) \in L^2(\mathbb{R}^n)$ the Wigner function $W\psi(q, p)$ (see [Wig]) is given by the expression

$$W\psi(q, p) = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^n} \mathrm{e}^{(2\mathrm{i}/\hbar)px} \psi(q - x)\overline{\psi(q + x)}\,\mathrm{d}x. \tag{2.2}$$

$W\psi(q, p)$ is the simplest probability function of the simultaneous values of q for the coordinates and p for the momenta. One has indeed

$$\int_{\mathbb{R}^n} W\psi(q, p)\,\mathrm{d}p = |\psi(q)|^2 \qquad \int_{\mathbb{R}^n} W\psi(q, p)\,\mathrm{d}q = |\widehat{\psi}(p)|^2$$

and, furthermore,

$$\int_{\mathbb{R}^{2n}} W\psi(q, p)\,\mathrm{d}p\,\mathrm{d}q = \|\psi\|_{L^2}^2.$$

Given now any two vectors $\psi, \varphi \in \mathcal{L}^2(\mathbb{R}^n)$, their *Wigner distribution* $W(\psi, \varphi)(q, p)$ is

$$W(\psi, \varphi)(q, p) = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{R}^n} \mathrm{e}^{(2\mathrm{i}/\hbar)px} \psi(q - x)\overline{\varphi(q + x)}\,\mathrm{d}x.$$

$W(\psi, \varphi)$ can be used to express the operators through their Weyl symbols (for these notions, see e.g. [Ber–Shu], section 5); in fact, as recalled in formula (1.2) above, the following result holds (for the proof, see e.g. [Ber–Shu], section 5.4):

Proposition 2.2. Given a quantum observable, i.e. an operator $\widehat{f}$ acting in $L^2(\mathbb{R}^n)$ with Weyl symbol $f(p,q)$ the matrix elements of $\widehat{f}$ are given by

$$\langle \widehat{f}\psi, \varphi\rangle_{\mathcal{L}^2} = \int W(\psi,\varphi)(p,q) f(p,q)\,\mathrm{d}p\,\mathrm{d}q. \tag{2.3}$$

The basic result about the Wigner function in the Bargmann representation is

Proposition 2.3. Let $F(z), G(z)$ be the unitary images in $\mathcal{F}_n$ of the vectors ψ, φ. Then:

(1) the Wigner distribution $W(F,G)(\omega)$ is given by

$$W(F,G)(\omega) = \frac{2^n}{(\pi\hbar)^{2n}} \int \mathrm{e}^{-(1/\hbar)(2|\omega|^2 - 2z\omega + |z|^2)} F(-z + 2\bar{\omega})\overline{G(z)}\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

(2) the Wigner function $WF \equiv W(F,F)$ is real, i.e.

$$W(F,G) = \overline{W(G,F)} \qquad \Longrightarrow WF = \overline{WF}$$

(3) it holds that

$$\int_{\mathbb{R}^{2n}} WF(\omega)\,\mathrm{d}\omega\,\mathrm{d}\bar{\omega} = \|F\|^2_{\mathcal{F}}$$

(4) if $\widehat{A}$ denotes the operator in $\mathcal{F}$ defined by the Weyl quantization of the symbol $A(\omega)$, its matrix elements between vectors $U, V \in \mathcal{F}$ have the following expression through their Wigner distribution:

$$\langle \widehat{A}U, V\rangle_{\mathcal{F}} = \int W(U,V)(\omega) A(\omega)\,\mathrm{d}\omega\,\mathrm{d}\bar{\omega}. \tag{2.4}$$

The Wigner functions in the complex variables for the harmonic oscillator eigenstates can be expressed in terms of the Laguerre polynomials $L_k^{(j)}$, defined for non-negative k and j by

$$L_k^{(j)} = \sum_{m=0}^{k} (-1)^m \frac{(k+j)!}{(k-m)!(j+m)!m!} x^m.$$

The one-dimensional, normalized eigenstates of the harmonic oscillator in the Bargmann representation are

$$\psi_k(z) = \frac{1}{\sqrt{k!}} \left(\frac{z}{\sqrt{\hbar}}\right)^k \qquad k = 0, 1, \ldots. \tag{2.5}$$

We can restrict our considerations to the one-dimensional case, since the eigenstates of the n-dimensional oscillators are just products of the above one-dimensional ones and the n-dimensional Wigner transform clearly preserves the product structure.

Proceeding exactly as in [Fol], section 2.1, and keeping track of all $\hbar$-dependent factors, we obtain:

Proposition 2.4. Let $W(\psi_k, \psi_j)(\omega)$ be the (normalized) Wigner transform of the harmonic oscillator eigenstates ψ_k, ψ_j. Then

$$W(\psi_k,\psi_j)(\omega) = \begin{cases} \dfrac{2}{\pi\hbar}\mathrm{e}^{-2|\omega|^2/\hbar}(-1)^k \dfrac{\sqrt{k!}}{\sqrt{j!}} \left(\dfrac{\sqrt{\hbar}}{2}\right)^{k-j} \omega^{j-k} L_k^{(j-k)}\left(\dfrac{4|\omega|^2}{\hbar}\right) & \text{for } j \geqslant k \\ \dfrac{2}{\pi\hbar}\mathrm{e}^{-2|\omega|^2/\hbar}(-1)^j \dfrac{\sqrt{j!}}{\sqrt{k!}} \left(\dfrac{\sqrt{\hbar}}{2}\right)^{j-k} \omega^{k-j} L_j^{(k-j)}\left(\dfrac{4|\omega|^2}{\hbar}\right) & \text{for } k \geqslant j. \end{cases}$$

3. Classical limit of Wigner function for the harmonic oscillator

It is known by the stationary phase approximation arguments (see [Ber] and [Ber–Bal]) that the Wigner function of any eigenstate is peaked along $H(\omega, \bar{\omega}) = E$, decays exponentially for $H(\omega, \bar{\omega}) > E$ and oscillates for $H(\omega, \bar{\omega}) < E$. In this section we make this result more precise and find its extension to the Wigner distribution of any two eigenstates by stating and proving the main result of this paper, namely formula (1.5) of section 1.

To compute the classical limit ($\hbar \to 0$, $k \to \infty$, $\hbar k = A$) of the Wigner distributions we consider their expression, given in the proposition 2.4, in terms of the Laguerre polynomials and use their asymptotic expansions.

The behaviour of Laguerre polynomials $L_k^{(m)}(x)$ when $k \to \infty$ and x is unrestricted has been investigated by several authors (see [Erd] or [Mag]) and can be summarized as follows.

Lemma 3.1. Divide the real axis into the following four distinct regions: (1) x near 0, (2) $0 < x < \nu$, (3) x near ν, (4) $x > \nu$, where $\nu = 4k + 2$.

The asymptotic behaviour of the Laguerre polynomials $L_k^{(m)}$ is thus obtained through the following expressions.

Case 1. If $\nu^{-1/3}x \to 0$ then

$$L_k^{(m)}(x) \approx \frac{\Gamma(k+m+1)}{k!}\left(\frac{\nu x}{4}\right)^{-m/2} \mathrm{e}^{x/2} J_m[(\nu x)^{1/2}]$$

where J_m is the Bessel function of m order.

Case 2. If $0 < \sigma < \pi/2$, $\nu\sigma^3 \to \infty$ and $\nu(\pi/2 - \sigma) \to +\infty$, $x = \nu \cos^2\sigma$ then

$$L_k^{(m)}(\nu\cos^2\sigma) \approx \frac{2(-1)^k(2\cos\sigma)^{-m}}{\sqrt{\pi\nu\sin 2\sigma}} \exp(\nu\cos^2\sigma/2)\left[\sin\left(\frac{\nu}{4}(2\sigma - \sin 2\sigma) + \frac{\pi}{4}\right)\right].$$

Case 3. If $\nu \to \infty$, $x - \nu = o(\nu^{3/5})$ then

$$L_k^{(m)}(x) \approx \frac{(-1)^k}{2^m 3} \mathrm{e}^{x/2}\left(\frac{\nu - x}{\nu}\right)^{1/2}\left[J_{-1/3}\left(\frac{\nu^{-1/2}}{3}(\nu - x)^{3/2}\right) + J_{1/3}\left(\frac{\nu^{-1/2}}{3}(\nu - x)^{3/2}\right)\right].$$

Case 4. If $\sigma > 0$, $\nu\sigma^3 \to \infty$ and $x = \nu\cosh^2\sigma$ then

$$L_k^{(m)}(\nu\cosh^2\sigma) \approx (-1)^k \exp\left(\frac{\nu}{2}\cosh^2\sigma\right)\frac{\exp[-(\nu/4)(\sinh 2\sigma - 2\sigma)]}{(2\cosh)^m\sqrt{\pi\nu\sinh 2\sigma}}.$$

Before stating the main result we further recall some well known relevant properties of the Bessel functions and an equally well known limiting theorem on integrals depending on a parameter.

Lemma 3.2. The Bessel function

$$J_\nu(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!\,\Gamma(\nu + m + 1)}\left(\frac{z}{2}\right)^{2m+\nu}$$

is an entire function of z for $\nu = 0, 1, \ldots$. It verifies the following properties.

(1) The following recurrence formulae that connect three contiguous functions hold:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu z^{-1} J_\nu(z)$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_\nu'(z).$$

(2) It holds that

$$\int z^{\nu+1} J_\nu(z)\,\mathrm{d}z = z^{\nu+1} J_{\nu+1}(z).$$

In particular,

$$\int z^{\nu+1} J_0(z)\, \mathrm{d}z = z J_1(z).$$

(3) The expansion of the 'Hankel' types for large argument and fixed order gives

$$J_\nu(z) \approx \left(\frac{\pi}{2} z\right)^{-1/2} \left[\cos\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right].$$

(4) The Bessel functions are many valued for $\nu \neq 0, 1, \ldots$. They are one valued for all points z of the principal branch $-\pi < \arg z < \pi$. The values at the points z not on the principal branch can be reduced to the principal ones by means of the relation

$$J_\nu(z\, \mathrm{e}^{\mathrm{i}m\pi}) = \mathrm{e}^{\mathrm{i}m\pi\nu} J_\nu(z).$$

(5) The following equality holds:

$$\int_0^\infty t^{\mu-1} J_\nu(at)\, \mathrm{d}t = 2^{\mu-1} a^{-\mu} \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu\right)}{\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\mu\right)} \qquad \text{where } -\operatorname{Re}\nu < \operatorname{Re}\mu < \tfrac{3}{2}.$$

In particular,

$$\int_0^\infty J_\nu(t)\, \mathrm{d}t = \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\right)}.$$

Lemma 3.3. Let $f(x, n)$ be positive and monotonicaly decreasing (as x increases, n fixed) and let $\int_a^\infty \phi(x)\, \mathrm{d}x < \infty$. Then

$$\lim_{n\to\infty} \int_a^{\lambda_n} f(x, n)\phi(x)\, \mathrm{d}x = \int_a^\infty g(x)\phi(x)\, \mathrm{d}x$$

provided that $\lim_{n\to\infty} \lambda_n = \infty$, that $\lim_{n\to\infty} f(x, n) = g(x)$ uniformly on compacts, and that there exists $0 < A < +\infty$ such that $f(a, n) < A\ \forall n$ (see [Bro] and [Wat]).

Corollary 3.1. Let f be a function of bounded variation on $[a, \lambda_n]$; if $f(x, n)$ tends pointwise to a constant C as $n \to \infty$ and $\int_a^\infty \phi(x)\, \mathrm{d}x$ converges then

$$\lim_{n\to\infty} \int_a^{\lambda_n} f(x, n)\phi(x)\, \mathrm{d}x = C \int_a^\infty \phi(x)\, \mathrm{d}x.$$

We can now state the main result given by formula (1.5) from which it follows that the classical limit in the sense of distribution of the Wigner function for the harmonic oscillator $Wf_k(|\omega|)$ behaves as a normalized delta function supported on the classical orbits $\omega\bar{\omega} = A$ where A is the classical action.

Recall that, see [Gel–Shi], if S is a C^∞ hypersurface of an open subset X of $\mathbb{R}^n$, the Euclidean structure of $\mathbb{R}^n$ induces on S a Riemannian structure, and denote by $\mathrm{d}\sigma_S$ the induced surface element on S. The Dirac delta function on S is the distribution $\delta_S \in \mathcal{D}'(X)$ defined by

$$\langle \delta_S, \varphi\rangle = \int_S \varphi(x)\, \mathrm{d}\sigma_S(x) \qquad \varphi \in C_0^\infty(X).$$

Then

Proposition 3.5. If $f(\omega) \in C_0^\infty(\mathbb{R}^2)$, $j - k = m$, m fixed then

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int\int W(\psi_k, \psi_j)(\omega) f(\omega)\, \mathrm{d}\omega\, \mathrm{d}\bar{\omega} = \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{d}\theta \qquad (3.1)$$

where $f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) \equiv f(\sqrt{A}\cos\theta, \sqrt{A}\sin\theta)$.

Proof. Changing to polar coordinates $|\omega|$, θ, we have

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int\int W(\psi_k, \psi_j)(\omega) f(\omega)\, \mathrm{d}\omega\, \mathrm{d}\bar{\omega}$$

$$= \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int_0^{2\pi}\int_0^{\infty} W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$

we can divide the integral into four parts according to the regions of validity of the asymptotic expansions of the Laguerre polynomials (see lemma (3.1)). Hence the above expression becomes

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \left\{ \int_0^{2\pi}\int_0^{\sqrt{A+\hbar/2}\sin\alpha(\hbar)} W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta \right.$$

$$+ \int_0^{2\pi}\int_{\sqrt{A+\hbar/2}\sin\alpha(\hbar)}^{\sqrt{A+\hbar/2}\cos\beta(\hbar)} W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$

$$+ \int_0^{2\pi}\int_{\sqrt{A+\hbar/2}\cos\beta(\hbar)}^{\sqrt{A+\hbar/2}\cosh\gamma(\hbar)} W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$

$$\left. + \int_0^{2\pi}\int_{\sqrt{A+\hbar/2}\cosh\gamma(\hbar)}^{\infty} W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta \right\}$$

where $\alpha(\hbar)$ tends to zero slower than $\hbar$ but faster than $\hbar^{1/3}$, as $\hbar \to 0$ and $\beta(\hbar)$, $\gamma(\hbar)$ tend to zero slower than $\hbar^{1/3}$ but faster than $\hbar^{1/5}$, at the same limit.

It will be shown that the only integral giving a non-vanishing contribution is the third one. This is in accordance with the delta function definition because only in the third integral are we in a small neighbourhood of the orbit $|\omega|^2 = A$.

We proceed to compute individually the four integrals and, without losing generality, we set from now on $k \leqslant j$. The same result can also be obtained in an analogous way when $k \geqslant j$. From proposition 2.4 it follows that if $k \leqslant j$ then

$$W(\psi_k, \psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) = \frac{2}{\pi\hbar}\,\mathrm{e}^{-2|\omega|^2/\hbar}(-1)^k \frac{\sqrt{k!}}{\sqrt{(k+m)!}}\left(\frac{\sqrt{\hbar}}{2}\right)^{-m} |\omega|^m\,\mathrm{e}^{\mathrm{i}m\theta} L_k^{(m)}\left(\frac{4|\omega|^2}{\hbar}\right). \quad (3.2)$$

Set

$$\nu = \frac{4A}{\hbar} + 2.$$

Integral 1. In the first integral we have

$$0 \leqslant |\omega| \leqslant \frac{\sqrt{\hbar}}{2}\nu^{1/2}\sin\alpha(\hbar)$$

thus the argument of the Laguerre polynomial in the Wigner function is such that

$$0 \leqslant \frac{4|\omega|^2}{\hbar} \leqslant \nu \sin^2(\alpha(\hbar)).$$

In this situation we are in case 1 of the lemma 3.1 because we have

$$0 \leqslant \nu^{-1/3} \frac{4|\omega|^2}{\hbar} \leqslant \nu^{2/3} \sin^2(\alpha(\hbar)) \xrightarrow{\hbar\to 0} 0.$$

It follows that

$$L_k^{(m)}\left(\frac{4|\omega|^2}{\hbar}\right) \approx \frac{(k+m)!}{k!}\left(\nu\frac{|\omega|^2}{\hbar}\right)^{-m/2} \mathrm{e}^{2|\omega|^2/\hbar}\; J_m\left(\left(\nu\frac{4|\omega|^2}{\hbar}\right)^{1/2}\right)$$

and as a consequence from 3.2 we obtain

$$W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) \approx \frac{2}{\pi\hbar}(-1)^k\frac{\sqrt{(k+m)!}}{\sqrt{k!}}\left(\frac{\nu}{4}\right)^{-m/2}\mathrm{e}^{\mathrm{i}m\theta} J_m\left(\left(\nu\frac{4|\omega|^2}{\hbar}\right)^{1/2}\right).$$

The classical limit of the first integral is

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int_0^{2\pi}\int_0^{(\sqrt{\hbar}/2)\nu^{1/2}\sin\alpha(\hbar)} W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$

$$= \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{2}{\pi\hbar}(-1)^k\frac{\sqrt{(k+m)!}}{\sqrt{k!}}\left(\frac{\nu}{4}\right)^{-m/2}\int_0^{2\pi}\int_0^{(\sqrt{\hbar}/2)\nu^{1/2}\sin\alpha(\hbar)}$$

$$\times \mathrm{e}^{\mathrm{i}m\theta} J_m\left(\left(\nu\frac{4|\omega|^2}{\hbar}\right)^{1/2}\right) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta.$$

With the change of variable $t = 2(\nu/\hbar)^{1/2}|\omega|$ the above limit becomes

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{(-1)^k}{2\pi\nu}\frac{\sqrt{(k+m)!}}{\sqrt{k!}}\left(\frac{\nu}{4}\right)^{-m/2}\int_0^{2\pi}\int_0^{\nu\sin\alpha(\hbar)} \mathrm{e}^{\mathrm{i}m\theta}\, t\, J_m(t)\, f\left(\frac{\sqrt{\hbar}}{2}\nu^{-1/2} t\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}t\,\mathrm{d}\theta.$$

Now

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{\sqrt{(k+m)!}}{\sqrt{k!}}\left(\frac{\nu}{4}\right)^{-m/2} = \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} k^{m/2}\left(\frac{\nu}{4}\right)^{-m/2} = 1$$

and

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{(-1)^k}{2\pi\nu}\int_0^{2\pi}\int_0^{\nu\sin\alpha(\hbar)} \mathrm{e}^{\mathrm{i}m\theta}\, t\, J_m(t)\, f\left(\frac{\sqrt{\hbar}}{2}\nu^{-1/2} t\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}t\,\mathrm{d}\theta = 0$$

provided we prove

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{(-1)^k}{2\pi\nu}\int_0^{\nu\sin\alpha(\hbar)} t\, J_m(t)\,\mathrm{d}t = 0. \tag{3.3}$$

Since $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and its arguments, in the integration interval, tend to $(0, 0)$, it follows that at the limit the first integral contribution vanishes.

To prove (3.3) we proceed by induction on m.

If $m = 0$, by assertion (2) of the lemma 3.2 we have

$$\lim_{\hbar\to 0} \frac{(-1)^{A/\hbar}}{2\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} t J_0(t)\, \mathrm{d}t = \lim_{\hbar\to 0} \frac{(-1)^{A/\hbar}}{2\pi\nu} [\nu\sin\alpha(\hbar) J_1(\nu\sin\alpha(\hbar))].$$

Replacing J_1 by its Hankel expansion (see assertion (3) of lemma 3.2) we obtain

$$\lim_{\hbar\to 0} \frac{1}{2\pi} (-1)^{A/\hbar} \sin\alpha(\hbar) \left[\frac{\pi}{2}\nu\sin\alpha(\hbar)\right]^{-1/2} \cos(\nu\sin\alpha(\hbar) - \tfrac{3}{4}\pi) = 0.$$

If $m = 1$, integrating by parts, by assertion (1) of lemma 3.2 we have

$$\lim_{\hbar\to 0} \frac{(-1)^{A/\hbar}}{2\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} t J_1(t)\, \mathrm{d}t = \lim_{\hbar\to 0} \frac{(-1)^{A/\hbar}}{2\pi\nu} \left\{ [-t J_0(t)]\,|_{t_1=0}^{t_2=\nu\sin\alpha(\hbar)} - \int_0^{\nu\sin\alpha(\hbar)} J_0(t)\, \mathrm{d}t \right\}.$$

Proceeding as in the $m = 0$ case we see that the first term on the right-hand side vanishes and since by assertion (5) of lemma 3.2 $\int_0^\infty J_0(t)\, \mathrm{d}t$ converges we also have

$$\lim_{\hbar\to 0} -\frac{(-1)^{A/\hbar}}{2\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} J_0(t)\, \mathrm{d}t = 0.$$

Now suppose that (3.3) holds up to m and prove it for $m+1$ using the well known relation $t J_{m+1}(t) = -t J_{m-1} + 2m J_m(t)$ which also follows by assertion (1) of lemma 3.2. We have

$$\lim_{\hbar\to 0} \frac{(-1)^{A/\hbar}}{2\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} t J_{m+1}(t)\, \mathrm{d}t$$
$$= \lim_{\hbar\to 0} -\frac{(-1)^{A/\hbar}}{2\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} t J_{m-1}(t)\, \mathrm{d}t + \lim_{\hbar\to 0} \frac{m(-1)^{A/\hbar}}{\pi\nu} \int_0^{\nu\sin\alpha(\hbar)} J_m(t)\, \mathrm{d}t.$$

The first limit is equal to zero by the induction hypothesis and the second one vanishes because

$$\int_0^{\nu\sin\alpha(\hbar)} J_m(t)\, \mathrm{d}t$$

converges as $\hbar \longrightarrow 0$.

Integral 2. In the second integral we have

$$\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\left(\frac{\pi}{2} - \alpha(\hbar)\right) \leqslant |\omega| \leqslant \frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\beta(\hbar)$$

where $\beta(\hbar)$ tends to zero slower than $\hbar^{1/3}$ and faster than $\hbar^{1/5}$, while $\alpha(\hbar)$ tends to zero slower than $\hbar$ and faster than $\hbar^{1/3}$.

Set

$$|\omega| = \frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\sigma \qquad \text{with} \qquad \beta(\hbar) \leqslant \sigma \leqslant \frac{\pi}{2} - \alpha(\hbar).$$

In this situation we are in case 2 of lemma 3.1. In fact,

$$\nu\beta^3(\hbar) \leqslant \nu\sigma^3 \leqslant \nu\left(\frac{\pi}{2} - \alpha(\hbar)\right)^3$$

hence $\nu\sigma^3 \to \infty$, and

$$\nu\alpha(\hbar) \leqslant \nu\left(\frac{\pi}{2} - \sigma\right) \leqslant \nu\left(\frac{\pi}{2} - \beta(\hbar)\right)$$

hence $\nu(\pi/2 - \sigma) \to \infty$.

By lemma 3.1 we have

$$L_k^{(m)}\left(\frac{4|\omega|^2}{\hbar}\right) = L_k^{(m)}\left(\nu\cos^2\sigma\right) \approx \frac{2(-1)^k(2\cos\sigma)^{-m}}{(\pi\nu\sin 2\sigma)^{1/2}}\exp\left(\nu\frac{\cos^2\sigma}{2}\right) \times \sin\left(\frac{\nu}{4}(2\sigma-\sin 2\sigma)+\frac{\pi}{4}\right)$$

and hence

$$W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) = W(\psi_k,\psi_j)\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\sigma\,\mathrm{e}^{\mathrm{i}\theta}\right) \approx \frac{4\sqrt{k!}}{\pi\hbar\sqrt{(k+m)!}}\left(\frac{\sqrt{\hbar}}{2}\right)^{-m}\frac{|\omega|^m\,\mathrm{e}^{\mathrm{i}m\theta}(2\cos\sigma)^{-m}}{(\pi\nu\sin 2\sigma)^{1/2}}\sin\left(\frac{\nu}{4}(2\sigma-\sin 2\sigma)+\frac{\pi}{4}\right).$$

The limit of the second integral is

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}}\int_0^{2\pi}\int_{(\sqrt{\hbar}/2)\nu^{1/2}\cos(\pi/2-\alpha(\hbar))}^{(\sqrt{\hbar}/2)\nu^{1/2}\cos\beta(\hbar)} W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$
$$= \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}}\int_0^{2\pi}\int_{\beta(\hbar)}^{\pi/2-\alpha(\hbar)}\frac{\sqrt{k!}}{\pi\sqrt{\pi\hbar}\sqrt{(k+m)!}}\left(\frac{1}{\sqrt{\hbar}}\right)^m\left(\frac{\hbar}{4}\nu\right)^{(m+1)/2} \times\mathrm{e}^{\mathrm{i}m\theta}(\sin 2\sigma)^{1/2}\frac{\sqrt{2}}{2}\left[\sin\left(\frac{\nu}{4}(2\sigma-\sin 2\sigma)\right)+\cos\left(\frac{\nu}{4}(2\sigma-\sin 2\sigma)\right)\right] \times f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\sigma\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\sigma\,\mathrm{d}\theta.$$

Putting $2\sigma-\sin 2\sigma=\phi$ the above limit becomes

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}}\frac{\sqrt{2}\sqrt{k!}}{4\pi\sqrt{\pi\hbar}\sqrt{(k+m)!}}\left(\frac{1}{\sqrt{\hbar}}\right)^m\left(\frac{\hbar}{4}\nu\right)^{(m+1)/2} \times\int_0^{2\pi}\int_{2\beta(\hbar)-\sin 2\beta(\hbar)}^{\pi-2\alpha(\hbar)-\sin(\pi-2\alpha(\hbar))}\frac{\sqrt{\sin 2\sigma(\phi)}}{1-\cos 2\sigma(\phi)}\mathrm{e}^{\mathrm{i}m\theta}\left[\sin\left(\frac{\nu}{4}\phi\right)+\cos\left(\frac{\nu}{4}\phi\right)\right] \times f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\sigma(\phi)\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\phi\,\mathrm{d}\theta.$$

Set

$$H(\phi,\theta) = \frac{\phi^{1/2}\sqrt{\sin 2\sigma(\phi)}}{1-\cos 2\sigma(\phi)}f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\sigma(\phi)\,\mathrm{e}^{\mathrm{i}\theta}\right).$$

Note that $H(\phi,\theta)$ is a function of bounded variation in $[0,\pi]$ (with respect to ϕ, uniformly with respect to θ). As a matter of fact, we have, as $\phi\to 0$, $\phi\approx\frac{4}{3}\sigma^3$ hence $\sigma\approx\left(\frac{3}{4}\phi\right)^{1/3}$ and

$$H(\phi,\theta)\approx\left[\sin((6\phi)^{1/3})\right]^{1/2}\phi^{1/2}\left[1-\cos((6\phi)^{1/3})\right]^{-1}f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) \approx 2(6\phi)^{1/6}\phi^{1/2}(6\phi)^{-2/3}f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) = \sqrt{\frac{2}{3}}\,f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta})$$

where the uniform boundedness of the ϕ derivative on $[0, \pi]$ follows by the de l'Hôpital rule. Hence the limit of the second integral becomes

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int_0^{2\pi} \int_{2\beta(\hbar)-\sin 2\beta(\hbar)}^{\pi-2\alpha(\hbar)-\sin(\pi-2\alpha(\hbar))} \frac{\sqrt{2}\sqrt{k!}}{4\pi\sqrt{\pi}\sqrt{(k+m)!}} \left(\frac{\nu}{4}\right)^{(m+1)/2} H(\phi,\theta)\phi^{-1/2}$$
$$\times \left[\sin\left(\frac{\nu}{4}\phi\right) + \cos\left(\frac{\nu}{4}\phi\right)\right] \mathrm{d}\phi\, \mathrm{d}\theta.$$

Now

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{\sqrt{k!}}{\sqrt{(k+m)!}} \left(\frac{\nu}{4}\right)^{m/2} = \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \left(\frac{1}{k}\right)^{m/2} \left(\frac{\nu}{4}\right)^{m/2} = 1$$

and by the change of variable $u = (1/4)\nu\phi$ the integration extrema for ϕ become

$$u_1(\hbar) = (1/4)\nu[2\beta(\hbar) - \sin 2\beta(\hbar)]$$
$$u_2(\hbar) = (1/4)\nu[\pi - 2\alpha(\hbar) - \sin(\pi - 2\alpha(\hbar))]$$

and the limit is

$$\lim_{\hbar\to 0} \frac{\sqrt{2}}{4\pi\sqrt{\pi}} \int_0^{2\pi} \int_{u_1(\hbar)}^{u_2(\hbar)} H\left(\frac{4u}{\nu}, \theta\right) u^{-1/2}(\sin u + \cos u)\, \mathrm{d}u\, \mathrm{d}\theta.$$

Now H is a function of bounded variation which tends to a constant depending only on θ as $\hbar \to 0$. Then, since both extrema of the integral $u_1(\hbar)$ and $u_2(\hbar)$ tend to ∞ and

$$\int_0^\infty u^{-1/2}(\sin u + \cos u)\, \mathrm{d}u$$

converges, by corollary 3.1 we have that the limit of the second integral also vanishes.

Integral 3. In the third integral

$$\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cos\beta(\hbar) \leqslant |\omega| \leqslant \frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\gamma(\hbar)$$

where $\beta(\hbar)$ and $\gamma(\hbar)$ tend to zero slower than $\hbar^{1/3}$ and faster than $\hbar^{1/5}$. Since

$$\nu[\cos^2\beta(\hbar) - 1] \leqslant \frac{4|\omega|^2}{\hbar} - \nu \leqslant \nu[\cosh^2\gamma(\hbar) - 1]$$

we are in case 3 of lemma 3.1. In fact

$$\nu^{2/5}\beta^2(\hbar) \leqslant \frac{1}{\nu^{3/5}}\left(\frac{4|\omega|^2}{\hbar} - \nu\right) \leqslant \nu^{2/5}\gamma^2(\hbar)$$

and hence as $\hbar \longrightarrow 0$

$$\frac{4|\omega|^2}{\hbar} - \nu = o(\nu^{3/5})$$

and

$$L_k^{(m)}\left(\frac{4|\omega|^2}{\hbar}\right) \approx \mathrm{e}^{2|\omega|^2/\hbar}\frac{(-1)^k}{2^m 3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\hbar}\right)^{1/2}$$
$$\times \left[J_{-1/3}\left(\frac{1}{3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\hbar}\right)^{3/2}\right) + J_{1/3}\left(\frac{1}{3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\hbar}\right)^{3/2}\right)\right].$$

The limit of the third integral is

$$\lim_{\substack{k\to\infty\\ \bar{h}\to 0\\ \bar{h}k=A}} \int_0^{2\pi} \int_{(\sqrt{\bar{h}}/2)\nu^{1/2}\cos\beta(\bar{h})}^{(\sqrt{\bar{h}}/2)\nu^{1/2}\cosh\gamma(\bar{h})} W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$

$$= \lim_{\substack{k\to\infty\\ \bar{h}\to 0\\ \bar{h}k=A}} \frac{2}{\pi\bar{h}} \int_0^{2\pi} \int_{(\sqrt{\bar{h}}/2)\nu^{1/2}\cos(\beta(\bar{h}))}^{(\sqrt{\bar{h}}/2)\nu^{1/2}\cosh(\gamma(\bar{h}))} \frac{1}{3} \frac{\sqrt{k!}}{\sqrt{(k+m)!}} (\sqrt{\bar{h}})^{-m} |\omega|^m\,\mathrm{e}^{\mathrm{i}m\theta}\nu^{-1/2}$$

$$\times \left(\nu - \frac{4|\omega|^2}{\bar{h}}\right)^{1/2} \left[J_{-1/3}\left(\frac{1}{3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\bar{h}}\right)^{3/2}\right)\right.$$

$$\left. + J_{1/3}\left(\frac{1}{3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\bar{h}}\right)^{3/2}\right)\right] f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta.$$

We make the substitution

$$t = \frac{1}{3}\nu^{-1/2}\left(\nu - \frac{4|\omega|^2}{\bar{h}}\right)^{3/2}.$$

With this change of variable the extrema of integration become

$$t_1 = \tfrac{1}{3}\nu(1-\cos^2\beta(\bar{h}))^{3/2} \qquad \text{and} \qquad t_2 = \tfrac{1}{3}\nu(1-\cosh^2\gamma(\bar{h}))^{3/2}.$$

Then the limit becomes

$$\lim_{\substack{k\to\infty\\ \bar{h}\to 0\\ \bar{h}k=A}} \frac{1}{6\pi} \int_0^{2\pi} \int_{t_2}^{t_1} \frac{\sqrt{k!}}{\sqrt{(k+m)!}} (\sqrt{\bar{h}})^{-m} \left[\frac{\bar{h}}{4}\left(\nu - (3t)^{2/3}\nu^{1/3}\right)\right]^{m/2} \mathrm{e}^{\mathrm{i}m\theta}$$

$$\times [J_{-1/3}(t) + J_{1/3}(t)] f\left(\left[\frac{\bar{h}}{4}\left(\nu - (3t)^{2/3}\nu^{1/3}\right)\right]^{1/2} \mathrm{e}^{\mathrm{i}\theta}\right) \mathrm{d}t\,\mathrm{d}\theta.$$

As $\bar{h} \longrightarrow 0$

$$t_1 \approx \tfrac{1}{3}\nu\beta^3(\bar{h}) \longrightarrow +\infty \qquad \text{and} \qquad t_2 \approx -\tfrac{1}{3}\nu\gamma^3(\bar{h}) \longrightarrow -\infty$$

and since $f \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and, as before,

$$\lim_{\substack{k\to\infty\\ \bar{h}\to 0\\ \bar{h}k=A}} \frac{\sqrt{k!}}{\sqrt{(k+m)!}} (\sqrt{\bar{h}})^{-m} \left[\frac{\bar{h}}{4}\left(\nu - (3t)^{2/3}\nu^{1/3}\right)\right]^{m/2} = 1$$

the limit expression becomes

$$\frac{1}{6\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) \int_{-\infty}^{+\infty} [J_{-1/3}(t) + J_{1/3}(t)]\,\mathrm{d}t\,\mathrm{d}\theta$$

$$= \frac{1}{6\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) \left\{ \int_{-\infty}^{0} [J_{-1/3}(t) + J_{1/3}(t)]\,\mathrm{d}t \right.$$

$$\left. + \int_0^{+\infty} [J_{-1/3}(t) + J_{1/3}(t)]\,\mathrm{d}t \right\} \mathrm{d}\theta$$

$$= \frac{1}{6\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta}) \left\{ \int_0^{+\infty} [J_{-1/3}(\mathrm{e}^{\mathrm{i}\pi}|t|) + J_{1/3}(\mathrm{e}^{\mathrm{i}\pi}|t|)]\,\mathrm{d}t \right.$$

$$\left. + \int_0^{+\infty} [J_{-1/3}(t) + J_{1/3}(t)]\,\mathrm{d}t \right\} \mathrm{d}\theta.$$

By assertion (4) of lemma 3.2 the above expression is equal to

$$\frac{1}{6\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\,\mathrm{e}^{\mathrm{i}\theta})\left\{\int_0^{+\infty}[\mathrm{e}^{-\mathrm{i}\pi/3}J_{-1/3}(|t|)+\mathrm{e}^{\mathrm{i}\pi/3}J_{1/3}(|t|)]\,\mathrm{d}t + \int_0^{+\infty}[J_{-1/3}(t)+J_{1/3}(t)]\,\mathrm{d}t\right\}\mathrm{d}\theta$$

and by assertion (5) of lemma 3.2 it becomes

$$\frac{1}{6\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\mathrm{e}^{\mathrm{i}\theta})\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}+\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}+2\right)\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}m\theta} f(\sqrt{A}\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta.$$

Integral 4. In the fourth integral we proceed as for the second one. In this case we have

$$\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\gamma(\hbar) \leqslant |\omega| \leqslant +\infty$$

where $\gamma(\hbar)$ tends to zero slower than $\hbar^{1/3}$ and faster than $\hbar^{1/5}$.

We put

$$|\omega| = \frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\sigma \qquad \text{with} \qquad \sigma > 0.$$

In the new variable σ, the integration interval becomes $[\gamma(\hbar),\infty]$ and we are in case 4 of lemma 3.1 because $\nu\sigma^3 \geqslant \nu\gamma^3(\hbar) \longrightarrow \infty$. Hence

$$L_k^{(m)}\left(\frac{4|\omega|^2}{\hbar}\right) = L_k^{(m)}\left(\nu\cosh^2\sigma\right) \approx (-1)^k \exp\left(\nu\frac{\cosh^2\sigma}{2}\right)\frac{\exp[-(1/4)\nu(\sinh 2\sigma - 2\sigma)]}{(2\cosh\sigma)^m[\pi\nu\sinh 2\sigma]^{1/2}}$$

and

$$W(\psi_k,\psi_j)\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\sigma\,\mathrm{e}^{\mathrm{i}\theta}\right) \approx \frac{2}{\pi\hbar}\frac{\sqrt{k!}}{\sqrt{(k+m)!}}\left(\frac{\nu}{4}\right)^{m/2}\mathrm{e}^{\mathrm{i}m\theta}\frac{\exp[-(1/4)\nu(\sinh 2\sigma - 2\sigma)]}{[\pi\nu\sinh 2\sigma]^{1/2}}.$$

The limit of the fourth integral is

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int_0^{2\pi}\int_{(\sqrt{\hbar}/2)\nu^{1/2}\cosh\gamma(\hbar)}^{+\infty} W(\psi_k,\psi_j)(|\omega|\,\mathrm{e}^{\mathrm{i}\theta}) f(|\omega|\,\mathrm{e}^{\mathrm{i}\theta})|\omega|\,\mathrm{d}|\omega|\,\mathrm{d}\theta$$
$$= \lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \int_0^{2\pi}\int_{\gamma(\hbar)}^{+\infty}\frac{1}{2\pi\sqrt{\pi}}\frac{\sqrt{k!}}{\sqrt{(k+m)!}}\left(\frac{\nu}{4}\right)^{(m+1)/2}\mathrm{e}^{\mathrm{i}m\theta}(\sinh 2\sigma)^{1/2}$$
$$\times\exp[-(1/4)\nu(\sinh 2\sigma - 2\sigma)] f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\sigma\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\sigma\,\mathrm{d}\theta.$$

Proceeding in the same way as in the computation of the limit of the second integral, we put $\phi = \sinh 2\sigma - 2\sigma$ and the above limit becomes

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{1}{4\pi\sqrt{\pi}}\int_0^{2\pi}\int_{\sinh(2\gamma(\hbar))-2\gamma(\hbar)}^{\infty}\mathrm{e}^{\mathrm{i}m\theta}\left(\frac{\nu}{4}\right)^{(m+1)/2}\frac{\sqrt{k!}}{\sqrt{(k+m)!}}\exp\left(-\frac{\nu}{4}\phi\right)$$
$$\times\frac{(\sinh 2\sigma(\phi))^{1/2}}{\cosh 2\sigma(\phi)-1} f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\sigma(\phi)\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\phi\,\mathrm{d}\theta.$$

Putting

$$H(\phi,\theta)=\frac{(\phi\sinh 2\sigma(\phi))^{1/2}}{\cosh 2\sigma(\phi)-1}\; f\left(\frac{\sqrt{\hbar}}{2}\nu^{1/2}\cosh\sigma(\phi)\,\mathrm{e}^{\mathrm{i}\theta}\right)$$

since

$$\lim_{\substack{k\to\infty\\ \hbar\to 0\\ \hbar k=A}} \frac{\sqrt{k!}}{\sqrt{(k+m)!}}\left(\frac{\nu}{4}\right)^{m/2}=1$$

we have to compute

$$\lim_{\hbar\to 0}\frac{\nu^{1/2}}{8\pi\sqrt{\pi}}\int_0^{2\pi}\int_{\sinh(2\gamma(\hbar))-2\gamma(\hbar)}^{+\infty}\exp\left(-\frac{\nu}{4}\phi\right)\phi^{-1/2}H(\phi,\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

We make the change of variable $u=(1/4)\nu\phi$ so that the above limit becomes

$$\lim_{\hbar\to 0}\frac{1}{4\pi\sqrt{\pi}}\int_0^{2\pi}\int_{(\nu/4)[\sinh 2\gamma(\hbar)-2\gamma(\hbar)]}^{\infty}\mathrm{e}^{-u}u^{-1/2}H\left(\frac{4}{\nu}u,\theta\right)\mathrm{d}u\,\mathrm{d}\theta.$$

Since H has a bounded variation, $H((4/\nu)u,\theta)$ tends to a constant depending only on θ,

$$\int_0^{+\infty}e^{-u}u^{-1/2}\,\mathrm{d}u$$

converges and

$$\tfrac{1}{4}\nu[\sinh 2\gamma(\hbar)-2\gamma(\hbar)]$$

tends to ∞, by corollary 3.1 the contribution of the fourth integral vanishes. This concludes the proof of the proposition. □

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